C2 Exercise 1D

Note Title 17/01/2007

1 Use the factor theorem to show

a) $(x-1)$ is a factor of $4x^3 - 3x^2 - 1$

The factor theorem states

$$(x-p) \text{ is a factor of } f(x) \Leftrightarrow f(p) = 0$$

find $f(1) = 4(1)^3 - 3(1)^2 - 1$

$= 4 - 3 - 1 = 0$

since $f(1) = 0$ then $(x-1)$ is a factor of $f(x)$

C2 Ex1D

1 Use the factor theorem to show

b) $(x+3)$ is a factor of $f(x) = 5x^4 - 45x^2 - 6x - 18$

find $f(-3) = 5(-3)^4 - 45(-3)^2 - 6(-3) - 18$

$= 5 \times 81 - 45 \times 9 + 18 - 18$

$= \cancel{405 - 405} + \cancel{18} - \cancel{18}$

$= 0$

$f(-3) = 0 \Leftrightarrow (x+3)$ is a factor of $f(x)$

C2 Ex 1D

1 Use the factor theorem to show

c) $(x-4)$ is a factor of $f(x) = -3x^3 + 13x^2 - 6x + 8$

find $f(4) = -3(4)^3 + 13(4)^2 - 6(4) + 8$

$= -3 \times 64 + 13 \times 16 - 24 + 8$

$= -192 + 208 - 24 + 8$

$= -216 + 216 = 0$

$f(4) = 0 \Leftrightarrow (x-4)$ is a factor of $f(x)$

C2 Ex1D

2 Show $(x-1)$ is a factor of $x^3+6x^2+5x-12$ and hence factorise the expression completely.

$$f(1) = (1)^3 + 6(1)^2 + 5(1) - 12 = 1+6+5-12 = 0$$

$$f(1)=0 \Leftrightarrow (x-1) \text{ is a factor of } f(x)$$

$$
\begin{array}{r|l}
 & x^2 + 7x + 12 \\
\hline
x-1 & x^3 + 6x^2 + 5x - 12 \\
 & -[x^3 - x^2] \\
\hline
 & 7x^2 + 5x \\
 & -[7x^2 - 7x] \\
\hline
 & 12x - 12 \\
 & 12x - 12 \\
\hline
 & 0
\end{array}
$$

Now

$$x^2+7x+12 = (x+3)(x+4)$$

So

$$x^3+6x^2+5x-12$$

$$\equiv (x-1)(x+3)(x+4)$$

C2 Ex1D

3 Show that $(x+1)$ is a factor of $x^3+3x^2-33x-35$ and hence factorise the expression completely.

$$f(-1) = (-1)^3 + 3(-1)^2 - 33(-1) - 35$$

$$= -1 + 3 + 33 - 35 = 36 - 36 = 0$$

$f(-1) = 0 \Leftrightarrow (x+1)$ is a factor of $f(x)$.

$$\begin{array}{r|l} & x^2 + 2x - 35 \\ \hline x+1 & x^3 + 3x^2 - 33x - 35 \\ & -[x^3 + x^2] \\ \hline & 2x^2 - 33x \\ & -[2x^2 + 2x] \\ \hline & -35x - 35 \\ & -[-35x - 35] \\ \hline & 0 \end{array}$$

Now $x^2 + 2x - 35 \equiv (x-5)(x+7)$

So $f(x) = (x+1)(x-5)(x+7)$

C2 Ex1D

4 Show that $(x-5)$ is factor of $x^3-7x^2+2x+40$ and hence factorise the expression completely.

$$f(5) = (5)^3 - 7(5)^2 + 2(5) + 40 = 125 - 175 + 10 + 40 = 0$$

$f(5)=0 \Leftrightarrow (x-5)$ is a factor of $f(x)$ by the factor theorem.

$$\begin{array}{r|l} & x^2 \quad - 2x \quad - 8 \\ \hline x-5 & x^3 - 7x^2 + 2x + 40 \\ & -[x^3 - 5x^2] \\ & \quad -2x^2 + 2x \\ & \quad -[-2x^2 + 10x] \\ & \quad\quad -8x + 40 \end{array}$$

Now $x^2-2x-8 = (x-4)(x+2)$

so $f(x) = (x-5)(x-4)(x+2)$

C2 Ex 1D

5 Show that $(x-2)$ is a factor of $2x^3+3x^2-18x+8$ and hence factorise the expression completely.

$$f(2) = 2(2)^3 + 3(2)^2 - 18(2) + 8 = 16 + 12 - 36 + 8 = 0$$

$f(2) = 0 \Leftrightarrow (x-2)$ is a factor of $f(x)$ by the factor theorem

$$
\begin{array}{r l}
 & 2x^2 + 7x - 4 \\
\cline{2-2}
x-2 \,) & 2x^3 + 3x^2 - 18x + 8 \\
 & -[2x^3 - 4x^2] \\
\cline{2-2}
 & 7x^2 - 18x \\
 & -(7x^2 - 14x) \\
\cline{2-2}
 & -4x + 8 \\
 & -(-4x + 8) \\
\cline{2-2}
 & 0
\end{array}
$$

Now $2x^2 + 7x - 4 = (2x-1)(x+4)$

So $f(x) = 2x^3 + 3x^2 - 18x + 8$

$= (x-2)(2x-1)(x+4)$

C2 Ex 1D

6a Given that $f(x) = x^3 - 10x^2 + 19x + 30$ has a factor $(x \pm p)$ find a suitable value of p and hence factorise $f(x)$.

$$f(-1) = (-1)^3 - 10(-1)^2 + 19(-1) + 30 = -1 - 10 - 19 + 30 = 0$$

so $f(-1) = 0 \Rightarrow (x+1)$ is a factor of $f(x)$ using factor theorem

$$\begin{array}{r|l} & x^2 - 11x + 30 \\ \hline x+1 & x^3 - 10x^2 + 19x + 30 \\ & -(x^3 + x^2) \\ \hline & \quad -11x^2 + 19x \\ & \quad -11x^2 - 11x \\ \hline & \qquad 30x + 30 \\ & \qquad -(30x + 30) \\ \hline & \qquad\qquad 0 \end{array}$$

Now $x^2 - 11x + 30 = (x-5)(x-6)$

So $f(x) = (x-6)(x-5)(x+1)$

You might have found $f(5)=0$ or $f(6)=0$ first and factorised different quadratics. That's fine, but it's a good idea to try $f(1)$ and $f(-1)$ first next time ...

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6b Given that $f(x) = x^3 + x^2 - 4x - 4$ has a factor $(x \pm p)$ find a suitable value of p and hence factorise $f(x)$.

try $x = 1$: $f(1) = (1)^3 + (1)^2 - 4(1) - 4 = 1 + 1 - 4 - 4 = -6$

try $x = -1$: $f(-1) = (-1)^3 + (-1)^2 - 4(-1) - 4 = -1 + 1 + 4 - 4 = 0$

$f(-1) = 0 \Leftrightarrow (x+1)$ is a factor of $f(x)$

$$\begin{array}{r l}
 & x^2 \qquad\quad - 4 \\
x+1 \,\big) & \overline{x^3 + x^2 - 4x - 4} \\
 & -\underline{(x^3 + x^2)} \\
 & \qquad\quad 0 - 4x - 4 \\
 & \qquad\quad -\underline{(-4x - 4)} \\
 & \qquad\qquad\qquad 0
\end{array}$$

Now $x^2 - 4 = (x+2)(x-2)$

so $f(x) = x^3 + x^2 - 4x - 4$

$= (x-2)(x+1)(x+2)$

C2 Ex1D

6c Given that $f(x) = x^3 - 4x^2 - 11x + 30$ has a factor $(x \pm p)$ find a suitable value of p and hence factorise $f(x)$.

You can see -1 & 1 won't work (30 is too big) so try -2 or 2

$$f(2) = 2^3 - 4(2)^2 - 11(2) + 30 = 8 - 16 - 22 + 30 = 38 - 38 = 0$$

$f(2) = 0 \Leftrightarrow (x-2)$ is a factor of $f(x)$

$$\begin{array}{r|rrrr}
 & x^2 & -2x & -15 & \\
\hline
x-2 & x^3 & -4x^2 & -11x & +30 \\
 & -(x^3 & -2x^2) & \downarrow & \downarrow \\
\hline
 & & -2x^2 & -11x & \\
 & & -(-2x^2 & +4x) & \downarrow \\
\hline
 & & & -15x & +30 \\
 & & & -(-15x & +30) \\
\hline
 & & & & 0
\end{array}$$

Now $x^2 - 2x - 15 = (x-5)(x+3)$

So $f(x) = x^3 - 4x^2 - 11x + 30$

$= (x-5)(x-2)(x+3)$

C2 Ex 1D

7a Factorise $2x^3 + 5x^2 - 4x - 3$

$f(1) = 2 + 5 - 4 - 3 = 0 \Rightarrow (x-1)$ is a factor of $f(x)$

$$
\begin{array}{r l}
 & 2x^2 + 7x + 3 \\
\cline{2-2}
x - 1 \,\big) & 2x^3 + 5x^2 - 4x - 3 \\
 & -(2x^3 - 2x^2) \\
\cline{2-2}
 & \quad 7x^2 - 4x \\
 & \quad -(7x^2 - 7x) \\
\cline{2-2}
 & \qquad 3x - 3 \\
 & \qquad -(3x - 3) \\
\cline{2-2}
 & \qquad\quad 0
\end{array}
$$

$2x^2 + 7x + 3 = (2x+1)(x+3)$

So $2x^3 + 5x^2 - 4x - 3 = (x-1)(2x+1)(x+3)$

C2 Ex 1D

7b factorise $f(x) = 2x^3 - 17x^2 + 38x - 15$

(odd powers positive, even negative $\Rightarrow p > 0$, else all terms < 0)

$f(1) = 2 - 17 + 38 - 15 = 40 - 32 = 8$

$f(2) = 2(2)^3 - 17(2)^2 + 38(2) - 15 = 16 - 68 + 76 - 15 = 92 - 83 = 9$

$f(3) = 2(3)^3 - 17(3)^2 + 38(3) - 15 = 54 - 153 + 114 - 15 = 168 - 168 = 0$

$$
\begin{array}{r|l}
 & 2x^2 - 11x + 5 \\
\hline
x - 3 & 2x^3 - 17x^2 + 38x - 15 \\
 & -(2x^3 - 6x^2) \\
\hline
 & -11x^2 + 38x \\
 & -(-11x^2 + 33x) \\
\hline
 & 5x - 15 \\
 & -(5x - 15) \\
\hline
 & 0
\end{array}
$$

$2x^2 - 11x + 5 = (2x - 1)(x - 5)$

$f(x) = (x - 5)(x - 3)(2x - 1)$

C2 Ex 1D

7c factorise $f(x) = 3x^3 + 8x^2 + 3x - 2$

$f(1) = 3 + 8 + 3 - 2 = 12$

$f(-1) = -3 + 8 - 3 - 2 = 0 \quad \Rightarrow (x+1)$ is a factor

$$
\begin{array}{r}
3x^2 + 5x - 2 \\
x+1 \overline{\smash{)}\,3x^3 + 8x^2 + 3x - 2} \\
-(3x^3 + 3x^2) \\
\hline
5x^2 + 3x \\
-(5x^2 + 5x) \\
\hline
-2x - 2 \\
-(-2x - 2) \\
\hline
0
\end{array}
$$

now $3x^2 + 5x - 2 = (3x - 1)(x + 2)$

so $f(x) = (3x-1)(x+1)(x+2)$

C2 Ex 1D

7d factorise $f(x) = 6x^3 + 11x^2 - 3x - 2$

$f(1) = 6 + 11 - 3 - 2 = 12$

$f(2) = 6(2)^3 + 11(2)^2 - 3(2) - 2 = 48 + 44 - 6 - 2 = 84$

$f(-1) = 6(-1)^3 + 11(-1)^2 - 3(-1) - 2 = -6 + 11 + 3 - 2 = 6$

$f(-2) = 6(-2)^3 + 11(-2)^2 - 3(-2) - 2 = -48 + 44 + 6 - 2 = 50 - 50 = 0$

$$
\begin{array}{r|l}
 & 6x^2 - x - 1 \\
\hline
x+2 & 6x^3 + 11x^2 - 3x - 2 \\
 & -(6x^3 + 12x^2) \\
\hline
 & -x^2 - 3x \\
 & -(-x^2 - 2x) \\
\hline
 & -x - 2
\end{array}
$$

Now $6x^2 - x - 1 = (3x+1)(2x-1)$

So $f(x) = (2x-1)(3x+1)(x+2)$

C2 Ex 1D

7e Factorise $f(x) = 4x^3 - 12x^2 - 7x + 30$

$f(1) = 4 - 12 - 7 + 30 = 34 - 19 = 15$

$f(2) = 4(8) - 12(4) - 7(2) + 30 = 32 - 48 - 14 + 30 = 62 - 62 = 0$

$$
\begin{array}{r|l}
 & 4x^2 - 4x - 15 \\
\hline
x - 2 & 4x^3 - 12x^2 - 7x + 30 \\
 & -(4x^3 - 8x^2) \\
\hline
 & -4x^2 - 7x \\
 & -(-4x^2 + 8x) \\
\hline
 & -15x + 30
\end{array}
$$

Now $4x^2 - 4x - 15 = (2x+3)(2x-5)$

So $f(x) = (2x-5)(x-2)(2x+3)$

C2 Ex 1D

8 Given that $(x-1)$ is a factor of $5x^3 - 9x^2 + 2x + a$, find a.

$(x-1)$ is a factor of $f(x) \Leftrightarrow f(1) = 0$

$$5(1)^3 - 9(1)^2 + 2(1) + a = 0$$

$$5 - 9 + 2 + a = 0$$

$$-2 + a = 0$$

$$a = 2$$

C2 Ex 1D

9 Given that $(x+3)$ is a factor of $6x^3 - bx^2 + 18$ find b.

$x+3$ is a factor $\Leftrightarrow$ $f(-3) = 0$

$\Rightarrow 6(-3)^3 - b(-3)^2 + 18 = 0$

$\Rightarrow -162 - 9b + 18 = 0$

$\Rightarrow 9b = -144$

$\Rightarrow b = -16$

C2 Ex 1D

10 Given that $(x-1)$ and $(x+1)$ are factors of px^3+qx^2-3x-7 find the value of p and q.

$f(1)=0$ since $(x-1)$ is a factor
$f(-1)=0$ since $(x+1)$ is a factor.

$f(1) = p+q-3-7 \Rightarrow p+q-10=0$ ①

$f(-1) = -p+q+3-7 \Rightarrow -p+q-4=0$ ②

① + ② $\Rightarrow 2q-14=0 \Rightarrow q=7$

$p+q-10=0 \Rightarrow p+7-10=0 \Rightarrow p=3$

$p=3,\ q=7.$